JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 20, No. 2, March–April 1997

Comparison of Inverse Modification Techniques for Some Gyroscopic Systems

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First- and second-order sensitivity and modification prediction techniques for gyroscopic systems are derived in order to develop effective inverse modification iterative procedures for such systems. Several techniques including inverse sensitivity methods, inverse modification techniques, and direct optimization methods are described and investigated. Accuracy and limitations of these algorithms are assessed by numerical studies of a rotating disk and a shaft rotor system. A comparison of these methods provides useful numerical information.

I. Introduction

A SIGNIFICANT task in system dynamics is the modification of system parameters to obtain dynamic behavior of a prescribed manner. Alternatively, the need to adjust system parameters to get a better agreement with test results may arise. Inverse modification techniques have been a research topic of interest because of their wide application in such problems. We classify inverse modification techniques into three groups: inverse sensitivity methods, inverse modification methods, and direct optimization methods.

Early investigations were mainly focused on inverse sensitivity methods and iterative modification techniques of undamped nongyroscopic systems. Bohte,[1] Tayor,[2] and Friedland et al.[3] investigated sensitivity methods for inverse eigenvalue problems. Linear perturbation methods, which allow for small structural changes, were introduced by Stetson and Palma,[4] and later Sandström and Anderson[5] described a similar inverse modification technique. Kim et al.[6] presented an inverse modification approach that included all nonlinear terms. However, its applicability is limited to small models because it requires a great deal of computation time. Kim and Anderson[7] employed dynamic condensation to reduce the problem size for large-scale systems. Hoff et al.[8] proposed a two-stage predictor–corrector algorithm that made it possible to predict large system changes. Bernitsas and Kang[9] used a multiple step modification technique to improve the predicted results with large modifications. Smith and Hutton[10] presented an improved inverse modification method utilizing baseline modes, which is more accurate for large modifications.

Gans and Anderson[11] employed the predictor–corrector method to optimize systems with Coriolis effects, which are formulated into a separate velocity-dependent matrix. Meirovitch[12] and Chen et al.[13] developed eigenvalue sensitivity analysis for gyroscopic systems.

The purpose of the present work is to formulate iterative equations based on the state equations of a gyroscopic system with multiple prescribed frequencies; to develop effective iterative procedures using sensitivity analysis, modification, and optimization techniques; and to investigate their accuracy and other characteristics. The comparison presented is designed to help researchers choose a suitable inverse modification technique that will work well for a specific gyroscopic system.

II. First- and Second-Order Sensitivity Analyses of Gyroscopic Systems

The equation of free vibration for a standard undamped gyroscopic system can be written as

$$\boldsymbol{M\ddot{u}} + \boldsymbol{G\dot{u}} + \boldsymbol{Ku} = \boldsymbol{0} \tag{1}$$

or in the first-order form

$$\boldsymbol{A\dot{x}} + \boldsymbol{Bx} = \boldsymbol{0} \tag{2}$$

where

$$\boldsymbol{x} = \begin{Bmatrix} \boldsymbol{\dot{u}} \\ \boldsymbol{u} \end{Bmatrix}, \qquad \boldsymbol{A} = \begin{bmatrix} \boldsymbol{M} & \boldsymbol{0} \\ 0 & \boldsymbol{K} \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{G} & \boldsymbol{K} \\ -\boldsymbol{K} & 0 \end{bmatrix}$$

$$(\boldsymbol{A} = \boldsymbol{A}^T,\ \boldsymbol{B} = -\boldsymbol{B}^T)$$

$\boldsymbol{M}$, $\boldsymbol{G}$, and $\boldsymbol{K}(M \times M)$ are the mass, gyroscopic, and stiffness matrices, respectively. In general, $\boldsymbol{M}$ is a real and symmetric matrix, $\boldsymbol{G}$ is a real and skew matrix, and $\boldsymbol{K}$ is real and symmetric but not necessarily positive definite. Assuming distinct eigenvalues, this kind of eigenproblem can be expressed by the following three equations:

$$\lambda_n \boldsymbol{A}\boldsymbol{x}_n + \boldsymbol{B}\boldsymbol{x}_n = 0 \tag{3}$$

$$\boldsymbol{x}_m^{*T}\boldsymbol{A}\boldsymbol{x}_n = \delta_{mn} \qquad \boldsymbol{x}_m^{*T}\boldsymbol{B}\boldsymbol{x}_n = -\delta_{mn}\lambda_n \tag{4}$$

where λ_n and $\boldsymbol{x}_n$ are the eigenvalue and eigenvector of the nth mode. Here, δ_{mn} is the Kronecker delta and superscripts T and $*T$ denote the transpose and complex conjugate transpose, respectively.

On the basis of these orthogonality properties and the assumption of purely imaginary eigenvalues (i.e., $\lambda_n = -\lambda_n^*$), we can derive the eigenvalue and eigenvector sensitivities with respect to a certain structural or physical parameter p. The eigenvalue and eigenvector sensitivities are given by

$$\lambda_{n,j} = -\left(\lambda_n \boldsymbol{x}_n^{*T}\boldsymbol{A}_{,j}\boldsymbol{x}_n + \boldsymbol{x}_n^{*T}\boldsymbol{B}_{,j}\boldsymbol{x}_n\right) \tag{5}$$

$$\boldsymbol{x}_{n,j} = \sum_{k=1}^{2N} a_{nk}\boldsymbol{x}_k \tag{6}$$

where

$$a_{nk} = \frac{1}{\lambda_k - \lambda_n}\boldsymbol{x}_k^{*T}(\lambda_n\boldsymbol{A}_{,p} + \boldsymbol{B}_{,p})\boldsymbol{x}_n \qquad (k \neq n) \tag{7}$$

$$a_{nn} = -\tfrac{1}{2}\boldsymbol{x}_n^{*T}\boldsymbol{A}_{,p}\boldsymbol{x}_n \qquad (k = n) \tag{8}$$

where subscript j following a comma indicates a partial derivative with respect to the jth design parameter. N can vary up to the problem order M, in which case the approximation becomes exact. It is important to note that Eqs. (5) and (6) are not true if the derivatives are calculated in a region where the eigenvalue is not purely imaginary.

Eigenvalues are usually nonlinear functions of system parameters and a second-order approximation offers a better prediction than

Received July 31, 1995; revision received Oct. 12, 1996; accepted for publication Nov. 13, 1996.

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the first-order approximation. Taking the derivative of Eq. (5) and simplifying the result, we obtain

$$\lambda_{n,jl} = -\left[\lambda_{n,l}\boldsymbol{x}_n^{*T}\boldsymbol{A}_{,j}\boldsymbol{x}_n + \boldsymbol{x}_{n,l}^{*T}\boldsymbol{Z}_n\boldsymbol{x}_n + \boldsymbol{x}_n^{*T}\boldsymbol{Z}_n\boldsymbol{x}_{n,l}^T + \boldsymbol{x}_n^{*T}(\lambda_n\boldsymbol{A}_{,jl} + \boldsymbol{B}_{,jl})\boldsymbol{x}_n\right] \tag{9}$$

where $\boldsymbol{Z}_n = \lambda_n\boldsymbol{A}_{,j} + \boldsymbol{B}_{,j}$. Subscripts jl following a comma indicate a second-order partial derivative with respect to the jth and lth design parameters. The change of the modal frequency corresponding to a set of parametric increments Δp_j can be expressed by use of a Taylor expansion as

$$\Delta\lambda_n = \sum_j \lambda_{n,j}\Delta p_j + \frac{1}{2}\sum_j\sum_l \lambda_{n,jl}\Delta p_j\Delta p_l + \mathcal{O}\left(\Delta p_j^3\right) \tag{10}$$

The eigenproblem and orthogonality properties of a general or damped gyroscopic system, which is non-self-adjoint, can be expressed by the following four equations:

$$\lambda_n\boldsymbol{A}\boldsymbol{x}_n + \boldsymbol{B}\boldsymbol{x}_n = 0, \qquad \lambda_n\boldsymbol{A}^T\boldsymbol{y}_n + \boldsymbol{B}^T\boldsymbol{y}_n = 0 \tag{11}$$

$$\boldsymbol{y}_m^T\boldsymbol{A}\boldsymbol{x}_n = \delta_{mn}, \qquad \boldsymbol{y}_m^T\boldsymbol{B}\boldsymbol{x}_n = -\delta_{mn}\lambda_n \tag{12}$$

where $\boldsymbol{x}_n$ and $\boldsymbol{y}_n$ are the right and left eigenvectors, respectively. It can be shown that the eigenvalue sensitivity of such a general gyroscopic system is given by

$$\lambda_{n,j} = -\boldsymbol{y}_n^T(\lambda_n\boldsymbol{A}_{,j} + \boldsymbol{B}_{,j})\boldsymbol{x}_n \tag{13}$$

III. Modification Prediction Techniques for Gyroscopic Systems

The eigenvalue equations of the unmodified and modified general gyroscopic systems can be expressed by the following four equations:

$$\boldsymbol{A}_0\boldsymbol{X}_0\Lambda_0 + \boldsymbol{B}_0\boldsymbol{X}_0 = \boldsymbol{0} \tag{14}$$

$$\boldsymbol{A}_0^T\boldsymbol{Y}_0\Lambda_0 + \boldsymbol{B}_0^T\boldsymbol{Y}_0 = \boldsymbol{0} \tag{15}$$

$$(\boldsymbol{A}_0 + \Delta\boldsymbol{A})\boldsymbol{X}\Lambda + (\boldsymbol{B}_0 + \Delta\boldsymbol{B})\boldsymbol{X} = \boldsymbol{0} \tag{16}$$

$$(\boldsymbol{A}_0 + \Delta\boldsymbol{A})^T\boldsymbol{Y}\Lambda + (\boldsymbol{B}_0 + \Delta\boldsymbol{B})^T\boldsymbol{Y} = \boldsymbol{0} \tag{17}$$

where $\boldsymbol{A}_0$ and $\boldsymbol{B}_0$ are the state matrices of the unmodified system with size $2M \times M$. $\Delta\boldsymbol{A}$ and $\Delta\boldsymbol{B}$ are the changes in the state matrices as a result of a system modification. Λ_0, $\boldsymbol{X}_0$, and $\boldsymbol{Y}_0$ are the full eigenvalue, right-, and left-eigenvector matrices of the original system, respectively. Λ, $\boldsymbol{X}$, and $\boldsymbol{Y}$ are the full modified eigenvalue, right-, and left-eigenvector matrices, respectively.

The eigenvalue matrix Λ_0 and the normalized mode shape matrices $\boldsymbol{X}_0$ and $\boldsymbol{Y}_0$ satisfy the following orthogonality relationships:

$$\boldsymbol{Y}_0^T\boldsymbol{A}_0\boldsymbol{X}_0 = \boldsymbol{I} \qquad \text{and} \qquad \boldsymbol{Y}_0^T\boldsymbol{B}_0\boldsymbol{X}_0 = -\Lambda_0 \tag{18}$$

Assume that the mode shapes for the modified system are linear combinations of the mode shapes of the unmodified system. That is,

$$\bar{\boldsymbol{X}} = \bar{\boldsymbol{X}}_0\boldsymbol{C} \qquad \text{and} \qquad \bar{\boldsymbol{Y}} = \bar{\boldsymbol{Y}}_0\boldsymbol{C}_L \tag{19}$$

where $\boldsymbol{C}$ and $\boldsymbol{C}_L$ are the matrices of unknown constants, which are also the modified right- and left-eigenvector matrices in mode space, respectively. Such a relationship is exact if $\bar{\boldsymbol{X}}_0 = \boldsymbol{X}_0$ or $\bar{\boldsymbol{Y}}_0 = \boldsymbol{Y}_0$. Substitution of Eq. (18) into Eqs. (16) and (17) and premultiplying by $\bar{\boldsymbol{Y}}_0^T$, which is the size-reduced left-eigenvector matrix, lead to a new eigenvalue problem in mode space:

$$\bar{\boldsymbol{A}}\boldsymbol{C}\bar{\Lambda} + \bar{\boldsymbol{B}}\boldsymbol{C} = \boldsymbol{0} \tag{20}$$

$$\bar{\boldsymbol{A}}^T\boldsymbol{C}_L\bar{\Lambda} + \bar{\boldsymbol{B}}^T\boldsymbol{C}_L = \boldsymbol{0} \tag{21}$$

where

$$\bar{\boldsymbol{A}} = \boldsymbol{I} + \bar{\boldsymbol{Y}}_0^T\Delta\boldsymbol{A}\bar{\boldsymbol{X}}_0 \qquad \text{and} \qquad \bar{\boldsymbol{B}} = -\bar{\Lambda}_0 + \bar{\boldsymbol{Y}}_0^T\Delta\boldsymbol{B}\bar{\boldsymbol{X}}_0 \tag{22}$$

$\bar{\Lambda}_0$ and $\bar{\Lambda}$ are the size-reduced eigenvalue matrices of the original and modified systems, respectively. The size of the eigenvalue problem in mode space is usually much less than that of the full unmodified model, and so the eigenvalue problem can be solved inexpensively for the modified mode shape matrices $\boldsymbol{C}$ and $\boldsymbol{C}_L$ and eigenvalue matrix $\bar{\Lambda}$. This procedure yields a satisfactory solution for the modified system if enough modes of the original system are contained in Eq. (18). Note that $\bar{\boldsymbol{Y}}_0 = \bar{\boldsymbol{X}}_0^*$ for a undamped gyroscopic system. In this case, the new normalized mode shapes in physical coordinates are then given by Eq. (18) if $\boldsymbol{C}$ is normalized by $\boldsymbol{C}^{*T}\bar{\boldsymbol{A}}\boldsymbol{C} = \boldsymbol{I}$.

IV. Inverse Modification Techniques for Gyroscopic Systems

An important aspect of system dynamics is the identification or adjustment of physical parameters to obtain improved dynamic behavior or get better agreement with test results. Six methods, as listed in Table 1, are described and discussed. The methods are classified into three groups: inverse sensitivity methods, inverse modification techniques, and direct optimization methods. Only the frequency-constrained gyroscopic problem is investigated.

A. Linear Inverse Sensitivity (IS-LIN) and Nonlinear Inverse Sensitivity (IS-NLIN) Methods

This class of inverse modification schemes is based on a series expansion about the baseline model. The frequency changes $\Delta\boldsymbol{f}(m \times 1)$ corresponding to a set of increments of system parameters $\Delta\boldsymbol{p}$ can be approximated by

$$\boldsymbol{S}_1\Delta\boldsymbol{p} + \tfrac{1}{2}\bar{\boldsymbol{S}}_2 = \Delta\boldsymbol{f} \tag{23}$$

where

$$\bar{\boldsymbol{S}}_2 = \left[\cdots\Delta\boldsymbol{p}^T\boldsymbol{S}_{2k}\Delta\boldsymbol{p}\cdots\right]^T$$

$\boldsymbol{S}_1$ and $\boldsymbol{S}_{2k}$ ($k = 1, 2, \ldots, m$) are the first- and second-order sensitivity matrices, respectively, and m is the number of eigenvalues to be modified.

Because the Newton–Raphson procedure (i.e., the linear sensitivity method) only involves first-order eigenvalue sensitivity, its iterative equation is given by

$$\left(\boldsymbol{S}_1^T W\boldsymbol{S}_1\right)\Delta\boldsymbol{p} = \boldsymbol{S}_1^T W\Delta\boldsymbol{f} \tag{24}$$

$$\boldsymbol{p}_L \le \boldsymbol{p}_0 + \Delta\boldsymbol{p} \le \boldsymbol{p}_U \tag{25}$$

where W is a diagonal weighting matrix used to emphasize specific mode frequencies or exclude poor quality test data and $\boldsymbol{p}_L$ and $\boldsymbol{p}_U$ are the bounds of the system parameters.

High-order sensitivities can also be used to improve the convergence of the iterative procedure. Thus,

$$\boldsymbol{S}_1\Delta\boldsymbol{p} + \tfrac{1}{2}\bar{\boldsymbol{S}}_2 + \mathcal{O}(\Delta\boldsymbol{p}^3) = \Delta\boldsymbol{f} \qquad \boldsymbol{p}_L \le \boldsymbol{p}_0 + \Delta\boldsymbol{p} \le \boldsymbol{p}_U \tag{26}$$

The inverse sensitivity procedure based on a least square minimization is briefly described as follows.

1) Choose an initial guess $\boldsymbol{p}_0$ for the system parameters.

2) Solve the eigenvalue problem of the gyroscopic system using the eigenvalue equation in physical or modal coordinates.

3) Calculate the sensitivity matrix of the system.

4) Calculate the system parameters satisfying Eq. (24), (25), or (26). When the solution is underdetermined, or the changes of

Table 1 Summary of inverse modification techniques for gyroscopic systems

Inverse sensitivity methods	
Linear sensitivity	(IS-LIN)
Nonlinear sensitivity	(IS-NLIN)
Inverse modification methods in mode space	
Inverse perturbation	(MS-INVP)
Linear inverse perturbation	(MS-LINVP)
Inverse sensitivity in modal space	(MS-INVS)
Direct optimization methods in mode space	(MS-DOPT)

parameters need to be limited, it is necessary to use an optimization algorithm.

5) Obtain the updated system parameters: $\boldsymbol{p} = \boldsymbol{p}_0 + \Delta\boldsymbol{p}$.

6) Repeat steps 2–5 until $\|\Delta\boldsymbol{p}\| \leq \varepsilon$, where ε is a small positive number.

B. Iterative Procedures Using Modification Techniques

The inverse sensitivity method is usually valid only for small parameter modifications in a single analysis of the baseline system. But the iterative procedure using modification techniques makes it possible to improve the predicted result before rerunning the full-size model. Three practical methods are described and discussed in this section.

1. Inverse Perturbation Method in Mode Space (MS-INVP)

The procedure proposed here is an extension of the method described by Smith and Hutton.[10] It is an inverse iterative modification approach using Eq. (20). For an undamped system, Eq. (20) can be rewritten as

$$\bar{\Lambda}_0\boldsymbol{C} - \boldsymbol{C}\bar{\Lambda} = \bar{\boldsymbol{X}}_0^{*T}\Delta\boldsymbol{A}\bar{\boldsymbol{X}}_0\boldsymbol{C}\bar{\Lambda} + \bar{\boldsymbol{X}}_0^{*T}\Delta\boldsymbol{B}\bar{\boldsymbol{X}}_0\boldsymbol{C} \tag{27}$$

where $\bar{\Lambda}_0(2N \times 2N)$ and $\bar{\Lambda}(m \times m)$ are the original and objective eigenvalue matrices, respectively. N is the number of baseline modes used in the modification, and m is the number of eigenvalues to be modified. $\boldsymbol{C}(2N \times m)$ is the new eigenvector matrix in modal coordinates. The modified mode shapes $\bar{\boldsymbol{X}}(2M \times m)$ can be expressed in terms of the original mode shapes $\bar{\boldsymbol{X}}_0(2M \times 2N)$ and $\boldsymbol{C}(2N \times m)$, that is, $\bar{\boldsymbol{X}} = \bar{\boldsymbol{X}}_0\boldsymbol{C}$.

The system changes $\Delta\boldsymbol{A}$ and $\Delta\boldsymbol{B}$ are described by the summation of all parameter changes:

$$\Delta\boldsymbol{A} = \sum_j \boldsymbol{A}_{,j}\Delta p_j \qquad \Delta\boldsymbol{B} = \sum_j \boldsymbol{B}_{,j}\Delta p_j \tag{28}$$

Substituting Eq. (28) into Eq. (27) gives

$$\sum_j \left(\bar{\boldsymbol{X}}_0^{*T}\boldsymbol{A}_{,j}\bar{\boldsymbol{X}}_0\boldsymbol{C}\bar{\Lambda} + \bar{\boldsymbol{X}}_0^{*T}\boldsymbol{B}_{,j}\bar{\boldsymbol{X}}_0\boldsymbol{C}\right)\Delta p_j = \bar{\Lambda}_0\boldsymbol{C} - \boldsymbol{C}\bar{\Lambda} \tag{29}$$

The solution of the preceding equation will provide the required parameter changes to meet the modal objective. The inverse perturbation method using modification is described as follows.

1) Set $\boldsymbol{C} = [\boldsymbol{I}\ \ \boldsymbol{0}]^T$, this initial assumption is that the modified mode shapes equal the baseline eigenvectors, i.e., $\bar{\boldsymbol{X}} = \bar{\boldsymbol{X}}_0$.

2) Solve Eq. (29) using only its diagonal terms to obtain the change of system parameters Δp_j and system matrices $\Delta\boldsymbol{A}$ and $\Delta\boldsymbol{B}$. When the solution is underdetermined or the parameter changes need to be limited, it is necessary to use an optimization algorithm.

3) Substitute the system matrix changes $\Delta\boldsymbol{A}$ and $\Delta\boldsymbol{B}$ into Eq. (27), resulting in the following iterative equation:

$$\boldsymbol{F}^{(i+1)} = \bar{\boldsymbol{X}}_0^{*T}\Delta\boldsymbol{A}\bar{\boldsymbol{X}}^{(i)}\bar{\Lambda} + \bar{\boldsymbol{X}}_0^{*T}\Delta\boldsymbol{B}\bar{\boldsymbol{X}}^{(i)} \tag{30}$$

where

$$\boldsymbol{F}^{(i+1)} = \bar{\Lambda}_0\boldsymbol{C}^{(i+1)} - \boldsymbol{C}^{(i+1)}\bar{\Lambda} \qquad \text{and} \qquad \bar{\boldsymbol{X}}^{(i)} = \bar{\boldsymbol{X}}_0\boldsymbol{C}^{(i)}$$

Superscript (i) indicates the current iteration. The corrected mode shapes $\boldsymbol{C}^{(i+1)}$ in modal space can be obtained by solving Eq. (30). The explicit expression for $\boldsymbol{C}^{(i+1)}$ in this procedure is given by

$$c_{nk} = \frac{\boldsymbol{x}_{0n}^{*T}(\lambda_k\Delta\boldsymbol{A} + \Delta\boldsymbol{B})\boldsymbol{x}_k}{\lambda_{0n} - \lambda_k} \qquad (k \neq n) \tag{31}$$

$$c_{nn} = \frac{\boldsymbol{x}_{0n}^{*T}(\lambda_n\Delta\boldsymbol{A} + \Delta\boldsymbol{B})\boldsymbol{x}_n}{\lambda_{0n} - \lambda_n} \qquad (k = n,\ \lambda_{0n} \neq \lambda_n) \tag{32}$$

where λ_{0n} and $\boldsymbol{x}_{0n}$ are the nth eigenvalue and eigenvector of the unmodified system, respectively. Here, λ_n and $\boldsymbol{x}_n$ are the nth prescribed eigenvalue and calculated eigenvector of the desired system, respectively. It may be noted that Eq. (31) is similar to Eq. (7) in form.

4) Substitute the corrected $\boldsymbol{C}^{(i+1)}$ into Eq. (29) to provide the corrected parameter changes. Repeat steps 2–4 until $\Delta p_j^{(i+1)} = \Delta p_j^{(i)}$.

By using the outlined procedure, an improved inverse modification result can be obtained in a single analysis of the baseline system.

2. Linear Inverse Perturbation Method in Mode Space (MS-LINVP)

The proposed method is an improved version of the predictor–corrector algorithm described by Hoff et al.[8] Using this method, the inverse modification can be done in a small-sized mode space. In the predictor phase, the system parameter changes are obtained through the first-order sensitivity method from Eq. (24) or Eq. (29) in which $\boldsymbol{C}$ is set to $[\boldsymbol{I}\ \ \boldsymbol{0}]^T$:

$$\sum_j \left(\bar{\boldsymbol{X}}_0^{*T}\boldsymbol{A}_{,j}\bar{\boldsymbol{X}}_0\bar{\Lambda} + \bar{\boldsymbol{X}}_0^{*T}\boldsymbol{B}_{,j}\bar{\boldsymbol{X}}_0\right)\Delta p_j = \bar{\Lambda}_0 - \bar{\Lambda} \tag{33}$$

In the corrector phase, the calculated first-order approximation to system changes is used to determine the first-order approximation of the mode shapes in modal space. Assume that the changes of mode shapes $\Delta\boldsymbol{X}$ are linear combinations of the mode shapes of the unmodified system. That is,

$$\Delta\boldsymbol{X} = \bar{\boldsymbol{X}}_0\boldsymbol{C}' \tag{34}$$

Using this relationship and Eq. (6) and neglecting the nonlinear terms, we can get the first-order approximation to $\boldsymbol{C}'$. Note that the changes of the nth mode shape in terms of the mode shape sensitivity $\boldsymbol{x}_{n,j}$ can be expressed as

$$\Delta\boldsymbol{x}_n = \sum_j \boldsymbol{x}_{n,j}\Delta p_j \tag{35}$$

Substituting Eq. (6) into Eq. (35) and rearranging them into matrix form give the following expression:

$$\Delta\boldsymbol{x}_n = \bar{\boldsymbol{X}}_0\left(\sum_j \boldsymbol{a}_n\Delta p_j\right) = \bar{\boldsymbol{X}}_0\boldsymbol{C}'_n \tag{36}$$

where

$$\boldsymbol{C}'_n = \sum_j \boldsymbol{a}_n\Delta p_j$$

where $\boldsymbol{a}_n$ is given by Eq. (7). Using Eq. (36) the modified mode shape matrix, where $\boldsymbol{a}_n$ $\bar{\boldsymbol{X}}(2M \times m)$ in physical coordinates can be written as

$$\bar{\boldsymbol{X}} = \bar{\boldsymbol{X}}_0\boldsymbol{C}_l \tag{37}$$

$$\boldsymbol{C}_l = [\boldsymbol{I}\quad \boldsymbol{0}]^T + [\boldsymbol{C}'_1\quad \cdots\quad \boldsymbol{C}'_m] \tag{38}$$

Substituting Eq. (38) into Eq. (29) and solving Eq. (29), we can get the corrected parameter changes.

The linear inverse perturbation method with updating baseline modes in mode space (MS-LINVP) can be briefly described as follows.

1) Set $\boldsymbol{C}_l = [\boldsymbol{I}\ \ \boldsymbol{0}]^T$ and solve Eq. (33) only using its diagonal terms to obtain the changes of system parameters Δp_j.

2) Calculate the mode shape matrix in modal coordinates $\boldsymbol{C}_l$ satisfying Eq. (38).

3) Substitute $\boldsymbol{C}_l$ into Eq. (29) and calculate the corrected parameter changes Δp_j with or without forcing orthogonality conditions on the objective modes.

4) Substitute the corrected parameter changes into Eq. (38) to obtain the corrected $\boldsymbol{C}_l$. Repeat steps 2–4 until Δp_j does not change.

5) Update the baseline modes using a modification technique.

6) Repeat steps 1–5 until $\|\Delta\boldsymbol{p}\| \leq \varepsilon$.

This procedure is equivalent to the predictor–corrector method proposed by Hoff and Bernitsas[8] if the baseline modes are updated by the linear modification method.

3. Inverse Sensitivity Method in Modal Space (MS-INVS)

This method is similar to the procedure described in Sec. IV.A, but it performs multiple iterations without reanalyzing the full-size model and updates the eigenvalues and mode shapes of the modified system in the modal subspace by means of the modification technique. This procedure is summarized as follows, where i indicates the number of the iteration.

1) Let $i = 1$ and choose the initial guesses $\boldsymbol{p}_0$ for the system parameters.

2) Solve the full-size eigenvalue problem in physical coordinates.

3) Calculate the sensitivity matrix of the system.

4) Calculate the parameter changes $\Delta\boldsymbol{p}^{(i)}$ satisfying Eq. (24) or Eq. (26).

5) Reanalyze the small-size eigenvalue problem in modal coordinates using the modification prediction technique; determine the modified eigenvalues and normalized eigenvectors in modal subspace, and then transform the eigenvectors into physical coordinates by using Eq. (19). The iterative equations can be expressed as follows:

$$\bar{\boldsymbol{A}}\boldsymbol{C}\bar{\Lambda} + \bar{\boldsymbol{B}}\boldsymbol{C} = \boldsymbol{0} \qquad \text{and} \qquad \boldsymbol{C}^{*T}\bar{\boldsymbol{A}}\boldsymbol{C} = \boldsymbol{I} \tag{39}$$

6) Set $i = i + 1$ and repeat steps 3–5 until $\|\Delta\boldsymbol{p}^{(i)}\| \leq \varepsilon$.

7) Calculate the total parameter changes:

$$\Delta\boldsymbol{p} = \sum_i \Delta\boldsymbol{p}^{(i)}$$

C. Direct Optimization Procedures in Mode Space (MS-DOPT)

Optimization techniques can also be applied to the inverse modification of eigenvalue problems in gyroscopic systems. Nonlinear programming techniques with constraints may be classified into two groups, i.e., direct search methods without using derivatives, such as Hooke and Jeeves' algorithm and the complex algorithm and gradient methods in which the derivatives of the objective function are used, for example, Rosen gradient projection method and the reduced gradient method.[14,15]

According to the authors' experience, direct search methods appear to be the most successful approach because their persistent adaptive nature allows them to converge when other methods lose efficiency, which makes them particularly valuable for optimization problems with multiple parameters and complex nonlinear behavior. Some modified gradient methods, such as the gradient projection method, are also very attractive because the eigenvalue and/or eigenvector sensitivity analysis can directly be used in the optimization procedure.[16] These kinds of methods tend to be rather fast if the procedure is successful.

The direct search methods usually need a large number of objective function calculations, and each objective function calculation also requires significant computation time. One approach to save computer time is to employ modification techniques to calculate the eigenvalue problem in modal space.

The following expressions are used for the frequency-constrained inverse modification problem.

Objective function:

$$\text{Optimize} \sum_j (f_i - \bar{f}_i)^2 \tag{40}$$

Constraints:

$$\text{Subject to} \quad \boldsymbol{p}_L \leq \boldsymbol{p} \leq \boldsymbol{p}_U$$

Objective function calculation:

$$\bar{\boldsymbol{A}}\boldsymbol{C}\bar{\Lambda} + \bar{\boldsymbol{B}}\boldsymbol{C} = \boldsymbol{0}$$

where f_i and $\bar{f}_i$ are the ith calculated and prescribed frequencies, respectively. A reliable inverse modification procedure similar in spirit to Hooke and Jeeves' algorithm is developed and compared with the other methods.

Inverse modification problems have quite complicated optimization surfaces and constraints, and the search for a global optimum may stop at a local optimal point. An effective solution approach may need several runs, using different starting points or different constraints.

V. Numerical Examples and Comparisons

A. Rotating Disk with In-Plane Residual Stresses

To demonstrate the capabilities of the methods, the analysis of two examples, a rotating clamped disk with in-plane residual stresses (Fig. 1) and a flexible rotor supporting a rigid disk (Fig. 2), were considered. Figure 1 shows a representation of a circular disk of inner radius a, outer radius b, and thickness h rotating with a constant angular velocity Ω. The disk has been cold rolled, inducing plastic strains in the region R_L–R_U. This process is used to optimize the lateral stiffness of circular saw blades. Outside this region the stresses can be determined in terms of radial stresses S_L and S_U acting on the boundaries of the cold rolled region.[17] Stresses S_L and S_U can be related to the rolling process; however, in the present case these stresses will be considered to be design variables. Note that, as defined by Schajer,[17] S_L and S_U are assigned negative signs when S_L and S_U acting on the region R_L–R_U are directed toward the outer rim and the center of the disk, respectively.

The transverse deflection u of a rotating disk measured with respect to the stationary reference frame (r, θ, u) may be approximated by a Galerkin solution in the form[18]

$$u(r, \theta, t) = \sum_{m=0}^{M}\sum_{n=0}^{N}[C_{mn}(t)\cos(n\theta) + S_{mn}(t)\sin(n\theta)]R_{mn}(r) \tag{41}$$

where $C_{mn}(t)$ and $S_{mn}(t)$ are unknown functions to be determined and $R_{mn}(r)$ are known functions chosen to satisfy the inner and outer bending boundary conditions of the disk.

Applying the Galerkin procedure leads to a type of gyroscopic equation of the following form:

$$\boldsymbol{M}\ddot{\boldsymbol{U}}_{cs} + \boldsymbol{G}(\Omega)\dot{\boldsymbol{U}}_{cs} + \boldsymbol{K}(\Omega^2)\boldsymbol{U}_{cs} = \boldsymbol{0} \tag{42}$$

where $\boldsymbol{K}(\Omega^2) = \boldsymbol{K}_0 - \boldsymbol{K}_1(\Omega^2)$ and the Galerkin vector $\boldsymbol{U}_{cs} = \{\cdots C_{mn} \cdots S_{mn} \cdots\}$.

The numerical results that follow were calculated for the case of a sawblade clamped at its inner radius and free at its outer radius with $a = 0.1333$ m, $b = 0.406$ m, $h = 2.998e{-}3$ m. The problem considered is that of modifying the internal stress distribution, which is completely defined in terms of S_L and S_U, to obtain a prescribed modification to the natural frequencies of the rotating saw.

Table 2 presents the stress configurations used for the analysis of two different speed cases: one at 500 rpm, where there is relatively little contribution of the gyroscopic term in Eq. (42), and another one at 3500 rpm, where the gyroscopic effect becomes important

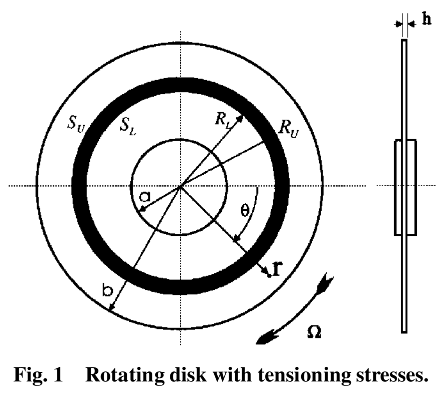

Fig. 1 Rotating disk with tensioning stresses.

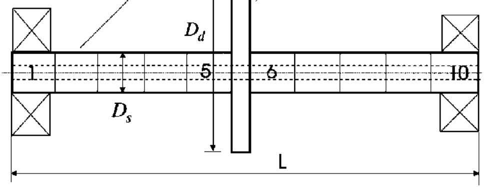

Fig. 2 Rotor system with central disk.

Table 2 Stress configurations considered

Cases	S_L, MPa	S_U, MPa	R_L, m	R_U, m	Rotating speed, rpm
Original A0	−10.0	−10.0	0.25	0.26	500
Modified A	−6.0	−7.0	0.25	0.26	500
Original B0	−24.0	−24.0	0.35	0.36	3500
Modified B	−5.0	−6.0	0.35	0.36	3500

Table 3 Comparison of modified and original frequencies

Mode	Original frequencies,[b] Hz		Modified frequencies, Hz		Frequency ratio	
number[a]	A0	B0	A	B	A/A0	B/B0
1	21.22(0, 0)	53.27(0, 0)	26.19	71.38	1.234	1.340
2	20.74(0, 1−)	15.55(0, 1−)	22.58	21.37	1.089	1.374
3	33.29(0, 2−)	132.2(0, 1+)	30.59	138.02	0.919	1.044

[a]Mode (M, $N\pm$) is the M nodal circles and N nodal diameters.
[b]Forward traveling wave + and backward traveling wave −.

Table 4 Inverse modification results: case A, four modes

Methods	S_L, MPa	S_U, MPa	E
MS-INVP	−5.5184	−7.2707	1.68e−2
MS-INVS	−5.4606	−7.2865	1.73e−2
MS-LINVP	−5.4550	−7.2896	1.77e−2
MS-DOPT	−5.4548	−7.2896	1.77e−2
IS-NLIN	−5.6609	−7.0383	1.80e−2
IS-LIN	−7.5167	−6.6839	2.11e−1

Table 5 Inverse modification results: case A, 10 modes

Methods	S_L, MPa	S_U, MPa	E
MS-INVS	−6.00000	−7.00000	2.17e−26
MS-DOPT	−6.00000	−7.00000	2.95e−15
MS-LINVP	−6.00000	−7.00000	7.13e−13
MS-INVP	−5.99999	−7.00000	3.71e−12

and $\boldsymbol{K}(\Omega^2)$ becomes negative definite. At each speed, two different stress distributions are specified, and three natural frequencies corresponding to each of these cases are given in Table 3. These results were obtained by solving Eq. (42) with model size $M = 2$ and $N = 20$. The results of this analysis were then used to investigate the inverse modification performance of the methods in attempting to predict the modified stresses presented in Table 2 in terms of a frequency modification specification as defined by Table 3.

The inverse modification results for case A using the six different methods discussed are given in Table 4. Four baseline modes of the original configurations were used in these procedures. In Table 4, the accuracy and iterative error

$$E = \sum_i (f_i - \bar{f}_i)^2$$

of these methods are examined. All of the methods give poor results, with IS-LIN performing worst of all. The reason for the poor performance was the small number of modes used. It was also found that IS-NLIN can improve the results obtained by IS-LIN to a certain extent.

To illustrate the effect of the number of baseline modes, the same problem was run with 10 baseline modes, and the results are presented in Table 5. All of the methods using modification techniques give almost exact results, with MS-INVS giving the best result. This result emphasizes the fact that sufficient baseline modes need to be included in the procedures to obtain an accurate result. It was also found that MS-DOPT has good accuracy and a persistent adaptive nature but it has a very slow convergence rate.

Figures 3 and 4 show the iterative error and inverse modification results vs the number of iterations using MS-INVP, MS-LINVP, and MS-INVS with 10 baseline modes. MS-INVS has the best accuracy

Table 6 Inverse modification results: case B, 10 modes

Methods	S_L, MPa	S_U, MPa	E
MS-INVS(a)	−5.00000	−6.00000	6.66e−27
MS-LINVP (a)	−4.99999	−6.00004	1.30e−12
MS-DOPT (a)	−4.99999	−6.00006	1.09e−11
MS-INVP (a)	−4.99994	−6.00034	4.75e−10
IS-NLIN (a)	−8.91890	−16.2835	2.07
IS-LIN (a)	−8.83602	−33.6671	74.1
IS-NLIN (b)	−4.98391	−6.09554	4.49e−5
IS-LIN (b)	−4.80260	−7.42130	1.67e−2

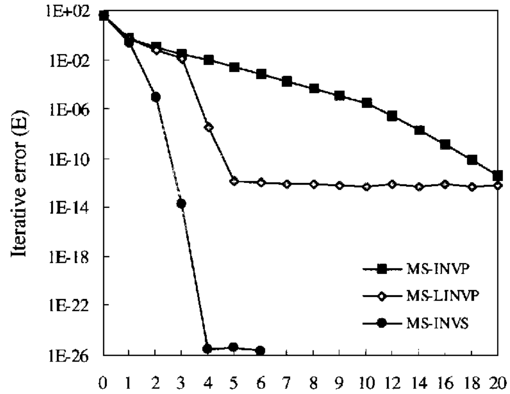

Fig. 3 Iterative error: case A, 10 modes.

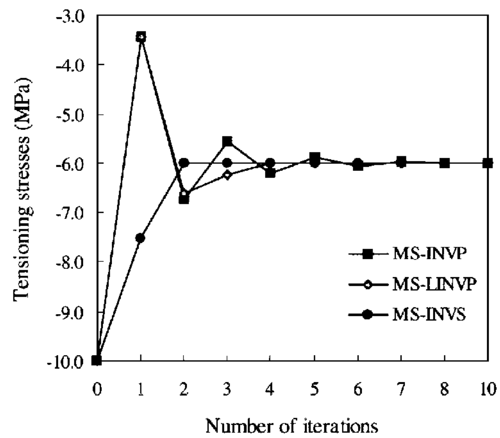

Fig. 4 Inverse modification results of S_L: case A, 10 modes.

and convergence rate. This is because of baseline mode updating using a modification technique at each iteration. In fact, it usually gives the best accuracy possible without reanalyzing the full-size model. MS-INVP and MS-LINVP work well but have a slower convergence rate.

To investigate the characteristics of the inverse modification methods as the gyroscopic effects become more important, cases Ba and Bb with both relatively poor and good initial guesses, case Ba $S_L = -24.0$ MPa, $S_U = -24.0$ MPa, and case Bb $S_L = 0.0$ MPa, $S_U = 0.0$ MPa, are examined, and the results are summarized in Table 6. All of the methods involving the modification technique in modal space give very good results. MS-INVS still gives results of highest accuracy and convergence rate when compared to MS-INVP, MS-LINVP, and MS-DOPT. The inverse sensitivity methods give poor results with a poor starting approximation but IS-NLIN gives an accurate result for case Bb, i.e., in the case where the initial guess is sufficiently close to the required solution.

To investigate the convergence behavior of the three typical methods, MS-INVP, MS-INVS, and MS-LINVP without rerunning the full-size model, cases Ba and Bb were examined. The results of case Ba are shown in Figs. 5 and 6. The convergence rate in MS-INVS

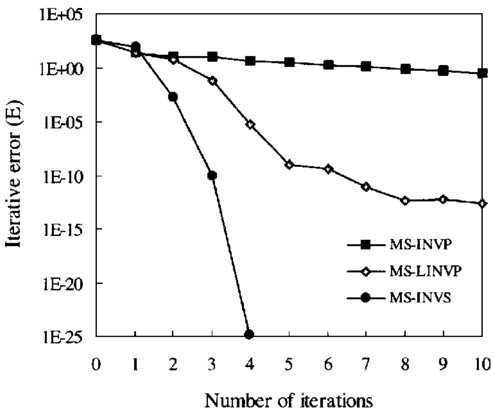

Fig. 5 Iterative error: case Ba, 10 modes.

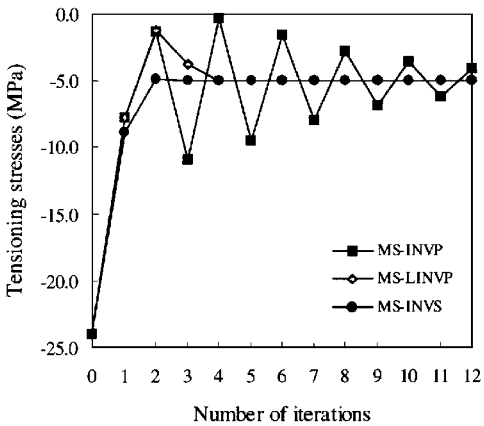

Fig. 6 Inverse modification results of S_L: case Ba, 10 modes.

is much faster than MS-INVP and the stress result from MS-INVP oscillates or even diverges with relatively poor starting guesses. For large parameter changes, MS-INVP has an approximately linear convergence rate whereas MS-INVS converges with a higher-order rate. The results of MS-LINVP are situated between those of MS-INVP and MS-INVS. The same conclusions were obtained from the results of case Bb.

Note that mode sequence in the equations may change during the iterative procedures, which can lead to divergence or poor accuracy. Therefore, it is necessary to keep the same mode sequence in the procedures.

B. Rotor with a Central Disk

The second example considers the case of a simply supported flexible rotating shaft with a rigid central disk rotating at a speed of 1500 rpm, as shown in Fig. 2. The shaft is made of steel with Young's modulus $E = 2.1e11$ N/m^2 and density $\rho = 7850$ kg/m^3. It has a length of 2 m, moment of inertia I_y equal to $4.91e{-}6$ m^4, and cross-sectional area A of $9.82e{-}6$ m^4. The thickness and inner and outer diameters of the central disk are 0.05, 0.1, and 0.3 m, respectively. One rigid disk element and 10 rotating beam elements are used, involving 40 degrees of freedom. The equation of motion for this type of undamped rotor system may be written as $\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{G}(\Omega)\dot{\boldsymbol{U}} + \boldsymbol{K}_0\boldsymbol{U} = \boldsymbol{0}$, where stiffness matrix $\boldsymbol{K}_0$ is independent of the rotating speed unlike the stiffness matrix of the flexible rotating disk.

The problem considered is that of raising the fundamental frequency ($f_1 = 62.2$ Hz) to 65.0 Hz by changing the moment of inertia in the sixth element. When 20 baseline modes are retained in the procedures, the modified variables and objective frequencies achieved in MS-INVP and MS-INVS are $7.3371e{-}6$ m^4, 64.986 Hz and $7.3476e{-}6$ m^4, 64.996 Hz, respectively. Figure 7 shows the iterative error vs the number of iterations with 10 and 20 baseline modes. A number of other cases with and without the central disk were also run to check the consistency of results. On the whole, conclusions consistent with those drawn from the analyses of a flexible rotating disk were obtained. The only difference was that MS-INVS

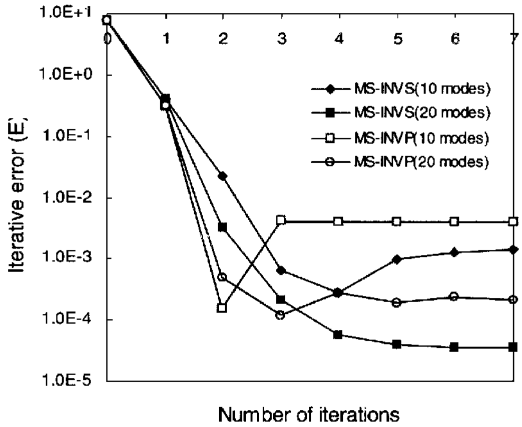

Fig. 7 Iterative error: rotor system, 10 and 20 modes.

converges rapidly only as more baseline modes are involved in the calculation. The objective frequency can be reached exactly and rapidly if 40 baseline modes are used in the procedure.

From a number of case studies, it is found that the three typical algorithms MS-INVS, MS-INVP, and MS-LINVP are about equivalent when few baseline modes are used and/or relatively small parameter changes are involved in the calculation. It is also found that the results of MS-LINVP are always situated between those of MS-INVP and MS-INVS.

VI. Discussion

The accuracy and efficiency of an inverse modification procedure depend mainly on its iteration algorithm and its strategy of baseline-mode updating. MS-INVS uses a Newton–Raphson iteration algorithm with a high convergence rate, whereas the iterative method employed in MS-INVP can be mathematically proved to be an algorithm with a slow convergence rate. It is the iterative method that may cause fluctuation or divergence for large-parameter changes.

Baseline-mode updating plays a key role in inverse modification procedures. The strategy used in the predictor–corrector methods proposed by Hoff et al.[8] is to update baseline modes using linear modification, which results in an inaccurate prediction of the modified modes shapes for large-parameter changes. A modification technique in modal space is employed to update baseline modes in MS-INVS and MS-LINVP, which retains all nonlinear coupling terms. The iterative procedure in MS-INVP is directly carried out within the characteristic equation in modal space instead of solving it for updating baseline modes, which slows the convergence rate.

It can be proved that the modification method involving eigenvalue analysis in modal space gives the best modified mode shapes possible with a given set of baseline modes. It can also be proved that the linear modifications with a large number of steps approach the accuracy equal to that of the modification method in modal space.[19] Therefore, it can be concluded that MS-INVS, MS-LINVP, or MS-INVP is more accurate than the predictor–corrector method, even the latter is used stepwise.[10] Note that the predictor–corrector method becomes equivalent to MS-INVS if the modification method in modal space is used to update baseline modes in the corrector phase.

Furthermore, the accuracy and convergence rate of an inverse modification procedure are also determined by the nonlinear characteristics of the eigenvalue curves with respect to design parameters, the difference between original and objective frequencies, and the number of baseline modes used in the calculation. For different problems, these factors will change the absolute accuracy and convergence rate of these procedures, but they will not change the relative accuracy among these algorithms if sufficient baseline modes are involved.

To obtain a more accurate solution with higher convergence rate, the full-size eigenvalue problem should be reanalyzed with the resulting parameter changes at each iteration or at an interval of several iterations. This scheme is specially suitable for small-scale problems.

VII. Conclusions

Some conclusions can be drawn from the preceding examples and analyses, which are summarized as follows.

1) In most cases, MS-INVS gives results with the best accuracy and efficiency when compared to other algorithms if sufficient baseline modes are used.

2) For large parameter changes, MS-INVP usually has a slow convergence rate.

3) The inverse modification procedures involving the characteristic equation in modal space, such as MS-INVP and MS-INVS, are more accurate and efficient than the predictor–corrector methods with linear baseline-mode update.

4) MS-LINVP may be considered a modified MS-INVS, whereas the predictor–corrector method may be considered to be a simplified MS-INVS with linear baseline-mode update.

5) IS-NLIN can be used to calculate the initial guesses of design parameters at the first iteration.

6) MS-DOPT usually requires a great deal of computation, but its adaptive nature enables it to be more reliable than other methods.

References

[1]Bohte, Z., "Numerical Solution of the Inverse Algebraic Eigenvalue Problem," *Computer Journal*, Vol. 10, No. 3, 1967, pp. 385–388.

[2]Taylor, J. E., "Scaling a Discrete Structural Modal to Match Measured Modal Frequencies," *AIAA Journal*, Vol. 15, No. 11, 1977, pp. 1647–1649.

[3]Friedland, S., Nocedal, J., and Overton, M. L., "The Formulation and Analysis of Numerical Methods for Inverse Eigenvalue Problems," *Journal of Numerical Analysis*, Vol. 24, No. 3, 1987, pp. 634–667.

[4]Stetson, K. A., and Palma, G. E., "Inversion of First-Order Perturbation Theory and Its Application to Structural Design," *AIAA Journal*, Vol. 14, No. 4, 1976, pp. 454–460.

[5]Sandström, R. E., and Anderson, W. J., "Modal Perturbation Methods for Marine Structures," *Society of Naval Architects and Marine Engineers Transactions*, Vol. 90, No. 1, 1982, pp. 41–54.

[6]Kim, K., Anderson, W. J., and Sandström, R. E., "Nonlinear Inverse Perturbation Method in Dynamic Analysis," *AIAA Journal*, Vol. 21, No. 9, 1983, pp. 1310–1316.

[7]Kim, K., and Anderson, W. J., "Generalized Dynamic Reduction in Finite Element Dynamic Optimization," *AIAA Journal*, Vol. 22, No. 11, 1984, pp. 1616, 1617.

[8]Hoff, C. J., Bernitsas, M. M., Sandström, R. E., and Anderson, W. J., "Inverse Perturbation Method for Structural Redesign with Frequency and Mode Shape Constraints," *AIAA Journal*, Vol. 22, No. 9, 1984, pp. 1304–1309.

[9]Bernitsas, M. M., and Kang, B., "Admissible Large Perturbation in Structural Redesign," *AIAA Journal*, Vol. 29, No. 1, 1991, pp. 104–113.

[10]Smith, M. J., and Hutton, S. G., "A Perturbation Method for Inverse Frequency Modification of Discrete, Undamped Systems," *Journal of Applied Mechanics*, Vol. 61, No. 4, 1994, pp. 887–892.

[11]Gans, H. D., and Anderson, W. J., "Structural Optimization Incorporating Centrifugal and Coriolis Effects," *AIAA Journal*, Vol. 29, No. 10, 1991, pp. 1743–1750.

[12]Meirovitch, L., "A New Method of Solution of the Eigenvalue Problem for Gyroscopic Systems," *AIAA Journal*, Vol. 12, No. 10, 1974, pp. 1337–1342.

[13]Chen, J. S., and Bogy, D. B., "Effects of Load Parameters on the Natural Frequencies and Stability of a Flexible Spinning Disk With a Stationary Load System," *Journal of Applied Mechanics*, Vol. 59, No. 2, 1992, pp. S230–S235.

[14]Himmelblau, D. M., *Applied Nonlinear Programming*, McGraw–Hill, New York, 1972, Chap. 3.

[15]Siddall, J. N., *Optimal Engineering Design: Principles and Applications*, Marcel Dekker, New York, 1982, Chaps. 2–4.

[16]Fiaco, A. V., *Introduction to Sensitivity and Stability Analysis in Non-Linear Programming*, Academic, New York, 1983, Chap. 4.

[17]Schajer, G. S., "Analysis of Roller-Induced Residual Stresses in Circular Discs and Their Effect on Disc Vibration," Ph.D. Dissertation, Dept. of Mechanical Engineering, Univ. of California, Berkeley, CA, Sept. 1981.

[18]Hutton, S. G., Chonan, S., and Lehman, B. F., "Dynamic Response of a Guided Circular Saw," *Journal of Sound and Vibration*, Vol. 112, No. 3, 1987, pp. 527–539.

[19]Tian, J., and Huang, S., "The Large-Change Sensitivity Analysis and Modification of the Dynamic Behaviors of Mechanical Structures," *Proceedings of 5th International Modal Analysis Conference* (London), 1987, pp. 1317–1324.